On Structural Controllability of Symmetric Brain Networks from One Region

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Objectives

1. Enrich theoretical analysis of structural controllability of anatomical connectivity networks from one node.
2. Develop and assess structural properties of networks with symmetry constraints on the choice of weights.

Abstract

The question of controllability of natural and man-made network systems has recently received considerable interest. In the context of the human brain, [1] has numerically shown that a class of brain networks constructed from DSI/DTI imaging data are controllable from one brain region, that is, a single brain region is theoretically capable of moving the whole brain network towards any desired target state. In this work we provide further evidence supporting controllability of brain networks from a single region.

Data: Anatomical Connectivity of 24 Subjects

State-of-the-art methods used to compute the matrices are able to track farther in to the lateral surfaces, and are also able to distinguish between more crossing fibers, enabling a fuller and more accurate assessment of the connectivity with respect to older, publicly available data.

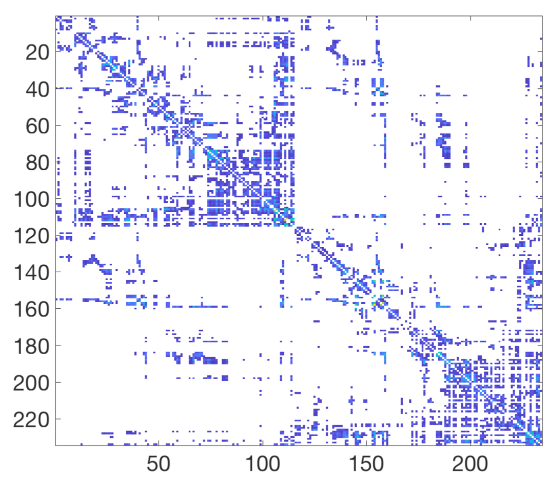

Figure 1: Connectivity matrix

$\Rightarrow$

DSI/DTI matrices

No diagonal entries and symmetry constraints

Notions from Structural Control Theory and Algebraic Geometry

- Network Dynamics:

$$x(t+1) = Ax(t) + b^{i}u(t) \tag{1}$$

The network is controllable if and only if the controllability matrix $\mathcal{C}(A, b^{i}) = \left[b^{i}\ Ab^{i} \cdots\ A^{n-1}b^{i}\right]$ is invertible.

- Structure Matrices: The elements of a structure matrix $A = [a_{ij}]$ are either fixed at zero or indeterminate values which are assumed to be <u>independent</u> of one another.

Determinant is a polynomial function in the entries of A

$$\det(\mathcal{C}(A, b^{i})) = \phi(a_{ij})$$
$$\Downarrow$$
If $\phi(a_{ij}) = 0$, controllability loss

Let $\mathcal{S}$ be the set of weights that render the network (1) not controllable:

$$\mathcal{S} = \{a_{ij}\ :\ (i,j) \in \mathcal{E} \text{ and } \phi(a_{ij}) = 0\}$$

$\mathcal{S}$ describes an algebraic variety of $\mathbb{R}^{d}$, where $d = |\mathcal{E}|$ is the number of nonzero entries of A.

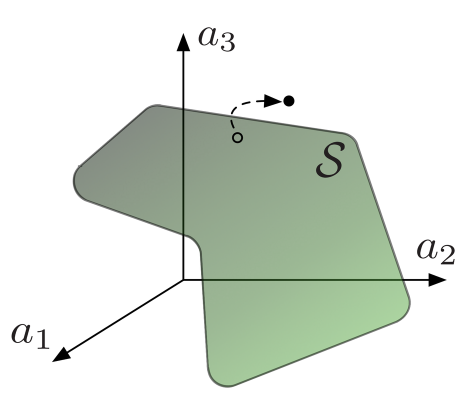

Figure 2: Small arbitrary perturbation of parameters makes the property hold

- This implies that controllability of (1) is a *generic property* because it fails to hold on a Zariski-closed subset of the parameters space [2]. Two mutually exclusive cases are possible:
  1. there exists *no* choice of weights a_{ij} rendering the network controllable.
  2. the network is controllable for all choices of weights a_{ij} except those lying in a proper algebraic variety of the parameters space.

  Thus, if one can find a choice of weights a_{ij} such that (1) is controllable, then *almost all* choices of weights a_{ij} yield a controllable network.
- If a property $\mathbf{\Pi}$ is generic relative to S and $\mathbf{\Pi}(\mathbf{p_0}) = \mathbf{0}$, a suitable perturbation of the parameters $\mathbf{p_0}$, which can be chosen arbitrarily small, can make the property hold.

Formal Discussion

Controllability of symmetric networks is a generic property

Let $d = |\mathcal{E}|$. The symmetry constraint on connectivity matrix' entries implies that the determinant of the controllability matrix is a polynomial function of $d/2$ parameters, because $a_{ij} = a_{ji}$ for all (i,j).

It is sufficient to construct one instance of a controllable symmetric network in order to show that almost all choices of weights yield a controllable network.

We construct a Hamiltonian path starting from the control node, select the weights of the edges in the path equal to one, and set all other weights equal to zero. Notice that the determinant of the controllability matrix associated with the constructed network has unit magnitude, proving that the network is structurally controllable [3].

Example

Structural controllability of a symmetric grid network by exploiting Hamiltonian paths:

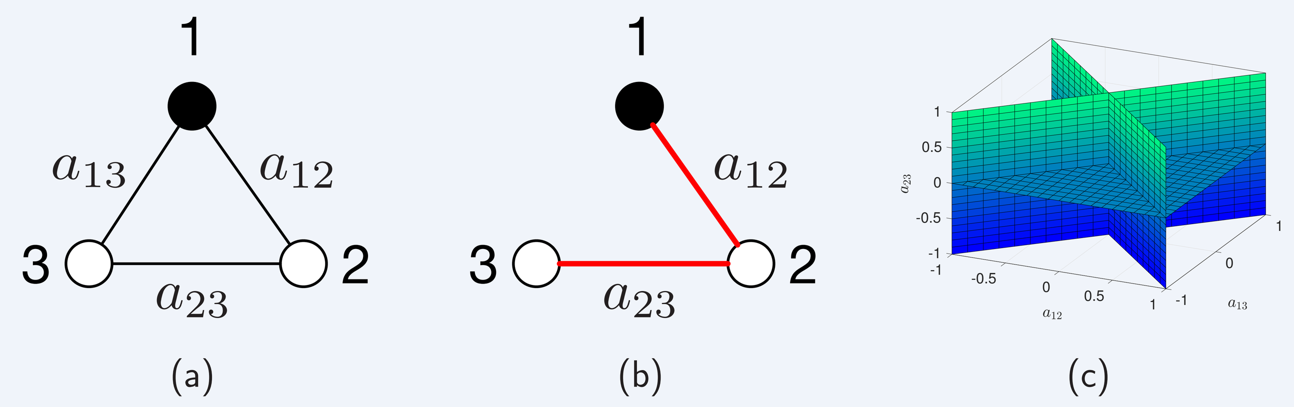

Figure 3: (a) Network with symmetric weights. (b) Network induced by a Hamiltonian path starting from the control node. (c) Algebraic variety containing the weights for which the network is not controllable. The network is controllable for all weights outside of this hypersurface.

$$|\det\left(\mathcal{C}(A, b^{i})\right)| = \mathbf{1} \text{ for all } i = \mathbf{1, 2, 3}$$

Results: Hamiltonian Paths on Structural Brain Networks

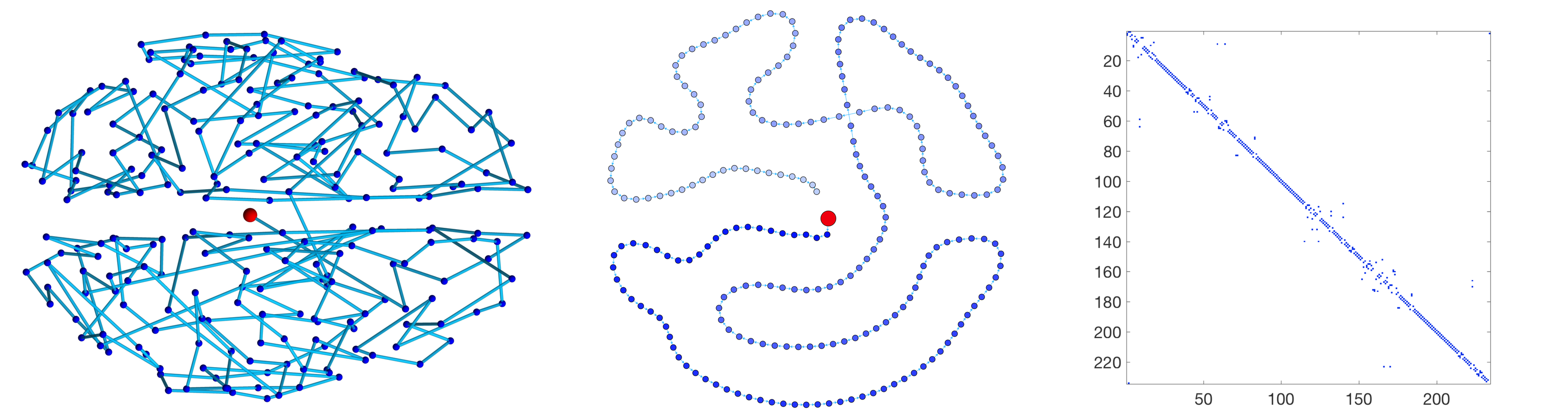

Figure 4: (a) Hamiltonian path in structural brain network of subject 1 from node 234, highlighted in red. (b) Hamiltonian path in two-dimensional visualization. (c) Adjacency matrix of the controllable realization with unitary weights.

Conclusion

- Network controllability is a generic property because it holds in the complement of the hypersurface of parameters that render the network uncontrollable. It is sufficient to show that one realization of a controllable symmetric network exists to overcome the constraint.
- The brain networks that we have analyzed admit a Hamiltonian path starting from every region, showing that these networks are structurally controllable even with symmetric weights.

References

[1] Shi Gu, Fabio Pasqualetti, Matthew Cieslak, Qawi K Telesford, B Yu Alfred, Ari E Kahn, John D Medaglia, Jean M Vettel, Michael B Miller, Scott T Grafton, and Danielle Bassett. Controllability of structural brain networks. *Nature Communications*, 6, 2015.

[2] David Mumford. *The red book of varieties and schemes: includes the Michigan lectures (1974) on curves and their Jacobians*, volume 1358. Springer Science & Business Media, 1999.

[3] Tommaso Menara, Shi Gu, Danielle Bassett, and Fabio Pasqualetti. On structural controllability of symmetric (brain) networks. *arXiv preprint*, 2017.

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