From Cluster Synchronization of Oscillators to Functional Connectivity

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**Model Motivation**

Functional connectivity in the human brain stems from synchronization of oscillatory neural activity in different brain regions.

We assume that at each node of the structural brain network exists a community of excitatory and inhibitory neurons whose dynamical state is in a regime of self-sustained oscillations.

**Sparsely connected heterogeneous Kuramoto oscillators**

\[ \dot{\theta}_i = \omega_i + \sum_{j \neq i} a_{ij} \sin(\theta_j - \theta_i) \]

\( \theta_i \) is the phase of the \( i \)-th oscillator, \( \omega_i \) is its natural frequency, and \( a_{ij} \) denotes the coupling strength between the \( j \)-th and the \( i \)-th oscillators.

**Thm 1**

The cluster synchronization manifold is stable

\[ |\{z\}| \leq 0 \]

**Problem Setup**

The brain can be abstracted by means of a network composed of nodes and edges:

- **structural brain networks**
  - nodes \( \leftrightarrow \) brain regions
  - edges \( \leftrightarrow \) white matter tracts that connect the regions

- **functional brain networks**
  - nodes \( \leftrightarrow \) brain regions
  - edges \( \leftrightarrow \) level of correlation between neurophysiological signals (e.g. fMRI)

Given a partition of the nodes \( P = \{P_1, \ldots, P_m\} \), we derive conditions that guarantee exponential stability of the cluster synchronization manifold

\[ S_P = \{\theta : \theta_i = \theta_j \text{ for all } i,j \in P_k, k = 1, \ldots, m\} \]

stable manifold \( S_P \), \( P = \{P_1, P_2, P_3\} \):

**Analytical Results**

**Result based on perturbation theory of dynamical systems:**

**Theorem 1:** sufficient condition on the network weights

The cluster synchronization manifold \( S_P \) is stable if:

- \( \alpha \) intra-cluster couplings
- \( \beta \) inter-cluster couplings

**Result based on Lyapunov’s stability theory:**

**Theorem 2:** sufficient conditions on the natural frequencies

The synchronization manifold \( S_P \) is stable if the frequencies across different clusters are sufficiently different

**Tighter Approximate Results**

Linearization of the network dynamics around cluster-synchronized trajectories + small gain theorem = tight stability conditions.

**Example**

\[ \alpha_1, \alpha_3 \leftrightarrow \text{intra-cluster coupling}, \quad \beta \leftrightarrow \text{inter-cluster coupling} \]

We fix \( \alpha_1 = \omega_1 = 1 \), \( \omega_2 = 8 \), and let \( \beta, \alpha_2 \) vary as indicated.

Fig. 1(a) is a network of 4 Kuramoto oscillators. Fig. 1(b) shows that the approximate stability condition is much tighter than the condition in Theorem 1. Our approximate conditions are able to tightly capture the onset of cluster synchronization.

**Conclusion and Future Work**

- **Exact stability conditions:** the cluster synchronization manifold is stable when the intra-cluster weights are sufficiently larger than the inter-cluster weights, and also when the natural frequencies across different clusters are sufficiently heterogeneous.

- **Approximate stability conditions:** combine the independent mechanisms of stability based on the network weights and oscillators’ natural frequencies, and are shown to be considerably more accurate than our exact conditions.

- **Future work:** validation of the clustering mechanisms on different datasets, prediction of clusters from connectivity information, data-driven control of cluster synchronization.

**References**
